

Alternative reverse inequalities for Young's inequality

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Abstract. Two reverse inequalities for Young's inequality were shown by M. Tominaga, using Specht ratio. In this short paper, we show alternative reverse inequalities for Young's inequality without using Specht ratio.

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1 Introduction

It is well known the Young inequality

$$a^{1-\lambda}b^\lambda \leq (1-\lambda)a + \lambda b, \quad (1)$$

for positive real numbers a, b and $\lambda \in [0, 1]$. See [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] for improvements of Young's inequality and their recent advances.

One of reverse inequalities for Young inequality was given by M. Tominaga in [11], using the Specht ratio, in the following way

$$(1-\lambda)a + \lambda b \leq S\left(\frac{a}{b}\right) a^{1-\lambda}b^\lambda, \quad (2)$$

for positive real numbers a, b and $\lambda \in [0, 1]$, where the Specht ratio [12, 13, 14], was defined by

$$S(h) \equiv \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}, \quad (h \neq 1)$$

for a positive real number h . Note that $\lim_{h \rightarrow 1} S(h) = 1$ and $S(h) = S(1/h) > 1$ for $h \neq 1, h > 0$. We call the inequality (2) a ratio-type reverse inequality for Young's inequality. M. Tominaga also showed in [11] the following inequality:

$$(1-\lambda)a + \lambda b \leq L(a, b) \log S\left(\frac{a}{b}\right) + a^{1-\lambda}b^\lambda, \quad (3)$$

for positive real numbers a, b and $\lambda \in [0, 1]$, where the logarithmic mean [15] $L(x, y)$ is defined by

$$L(x, y) \equiv \frac{y-x}{\log y - \log x}, \quad (x \neq y), \quad L(x, x) = x.$$

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We call the inequality (3) a difference-type reverse inequality for Young's inequality. Based on the scalar inequalities (2) and (3), M. Tominaga showed the following two reverse inequalities for invertible positive operators.

Theorem 1.1 ([11]) *For invertible positive operators A and B with $0 < mI \leq A, B \leq MI$, we have*

(i) *(Ratio-type reverse inequality)*

$$(1 - \lambda)A + \lambda B \leq S(h)A\sharp_{\lambda}B. \quad (4)$$

(ii) *(Difference-type reverse inequality)*

$$(1 - \lambda)A + \lambda B \leq A\sharp_{\lambda}B + L(1, h) \log S(h)B. \quad (5)$$

Our purpose of this short paper is to give two reverse inequalities which are different from (4) and (5).

2 Main results

We first show the following remarkable scalar inequality.

Theorem 2.1 *Let $f : [a, b] \rightarrow \mathbb{R}$ be a twice differentiable function such that there exist real constant M so that $0 \leq f''(x) \leq M$ for $x \in [a, b]$. Then we have*

$$0 \leq (1 - \lambda)f(a) + \lambda f(b) - f((1 - \lambda)a + \lambda b) \leq \lambda(1 - \lambda)M(b - a)^2, \quad (6)$$

where $\lambda \in [0, 1]$.

Proof: The first part of inequality (6) holds because f is a convex function. Next, we prove second part of inequality (6).

For $\lambda \in \{0, 1\}$, we obtain the equality in relation (6). Now, we consider $\lambda \in (0, 1)$, which means that $a < (1 - \lambda)a + \lambda b < b$ and we use Lagrange's theorem for the intervals $[a, (1 - \lambda)a + \lambda b]$ and $[(1 - \lambda)a + \lambda b, b]$. Therefore, there exists real numbers $c_1 \in (a, (1 - \lambda)a + \lambda b)$ and $c_2 \in ((1 - \lambda)a + \lambda b, b)$ such that

$$f((1 - \lambda)a + \lambda b) - f(a) = \lambda(b - a)f'(c_1) \quad (7)$$

and

$$f(b) - f((1 - \lambda)a + \lambda b) = (1 - \lambda)(b - a)f'(c_2). \quad (8)$$

Multiplying relation (7) by $(1 - \lambda)$ and relation (8) by λ , and then adding, we deduce the following relation:

$$(1 - \lambda)f(a) + \lambda f(b) - f((1 - \lambda)a + \lambda b) = \lambda(1 - \lambda)(b - a)[f'(c_2) - f'(c_1)].$$

Again, applying Lagrange's theorem on the interval $[c_1, c_2]$, we obtain

$$(1 - \lambda)f(a) + \lambda f(b) - f((1 - \lambda)a + \lambda b) = \lambda(1 - \lambda)(b - a)(c_2 - c_1)f''(c), \quad (9)$$

where $c \in (c_1, c_2)$. Since $0 \leq f''(x) \leq M$ for $x \in [a, b]$ and $c_2 - c_1 \leq b - a$ and making use of relation (9), we obtain the inequality (6). ■

Corollary 2.2 For $a, b > 0$ and $\lambda \in [0, 1]$, the following inequalities hold.

(i) (Ratio-type reverse inequality)

$$a^{1-\lambda}b^\lambda \leq (1-\lambda)a + \lambda b \leq a^{1-\lambda}b^\lambda \exp \left\{ \frac{\lambda(1-\lambda)(a-b)^2}{d_1^2} \right\}, \quad (10)$$

where $d_1 \equiv \min \{a, b\}$.

(ii) (Difference-type reverse inequality)

$$a^{1-\lambda}b^\lambda \leq (1-\lambda)a + \lambda b \leq a^{1-\lambda}b^\lambda + \lambda(1-\lambda) \left\{ \log \left(\frac{a}{b} \right) \right\}^2 d_2, \quad (11)$$

where $d_2 \equiv \max \{a, b\}$.

Proof:

(i) It is easy to see that if we take $f(x) = -\log x$ in Theorem 2.1, then we have

$$\log \{(1-\lambda)a + \lambda b\} \leq \log (a^{1-\lambda}b^\lambda) + \log (\exp \{ \lambda(1-\lambda)f''(c)(b-a)^2 \})$$

which implies inequality (10), since $f''(c) = \frac{1}{c^2} \geq \frac{1}{d_1^2}$.

(ii) If we take $f(x) = e^x$ (which is convex on $(-\infty, \infty)$) in Theorem 2.1, we obtain the relation

$$0 \leq \lambda e^\alpha + (1-\lambda)e^\beta - e^{\lambda\alpha + (1-\lambda)\beta} \leq \lambda(1-\lambda)(\alpha - \beta)^2 f''(\gamma),$$

where $\gamma \equiv \max \{\alpha, \beta\}$. Putting $a = e^\alpha$ and $b = e^\beta$, then we have

$$0 \leq (1-\lambda)a + \lambda b - a^{1-\lambda}b^\lambda \leq \lambda(1-\lambda)e^c \left(\log \frac{a}{b} \right)^2$$

where $c \equiv \max \{\log a, \log b\}$. Thus we have inequality (11), putting $d_2 = e^c$. ■

From here, we consider bounded linear operators acting on a complex Hilbert space \mathcal{H} . If a bounded linear operator A satisfies $A = A^*$, then A is called a self-adjoint operator. If a self-adjoint operator A satisfies $\langle x | A | x \rangle \geq 0$ for any $|x\rangle \in \mathcal{H}$, then A is called a positive operator. In addition, $A \geq B$ means $A - B \geq 0$.

Theorem 2.3 For $\lambda \in [0, 1]$, two invertible positive operators A and B satisfying the ordering $mI \leq A \leq B \leq MI \leq I$ with $h \equiv \frac{M}{m}$, we have the following operator inequalities.

(i) (Ratio-type reverse inequality)

$$A \sharp_\lambda B \leq (1-\lambda)A + \lambda B \leq \exp \left(\lambda(1-\lambda) \left(1 - \frac{1}{h} \right)^2 \right) A \sharp_\lambda B. \quad (12)$$

(ii) (Difference-type reverse inequality)

$$A \sharp_\lambda B \leq (1-\lambda)A + \lambda B \leq A \sharp_\lambda B + \lambda(1-\lambda) (\log h)^2 B. \quad (13)$$

Proof:

- (i) The first inequalities in (12) and (13) are well-known so that we prove the two second inequalities in (12) and (13). From the inequality (10) with $a \leq b$, we have

$$(1 - \lambda)t + \lambda \leq t^{1-\lambda} e^{\lambda(1-\lambda)(1-\frac{1}{t})^2},$$

for $0 < t \leq 1$. Thus we have the following inequality for the invertible positive operator $mI \leq T \leq MI \leq I$:

$$(1 - \lambda)T + \lambda \leq T^{1-\lambda} \max_{m \leq t \leq M} e^{\lambda(1-\lambda)(1-\frac{1}{t})^2}.$$

Putting $T \equiv B^{-1/2}AB^{-1/2} \leq I$ (which satisfies $A \leq B$), then we have $\frac{1}{h} \leq B^{-1/2}AB^{-1/2} \leq h$, and then we have

$$(1 - \lambda)B^{-1/2}AB^{-1/2} + \lambda \leq \left(B^{-1/2}AB^{-1/2}\right)^{1-\lambda} \max_{\frac{1}{h} \leq t \leq h} e^{\lambda(1-\lambda)(1-\frac{1}{t})^2}.$$

Multiplying $B^{1/2}$ to the both sides in the above inequality, we obtain the inequality (12), since $A\sharp_{\lambda}B = B\sharp_{1-\lambda}A$.

- (ii) By the similar way to the proof of the second inequality in (12) from the inequality (11) with $1 \leq a \leq b$, we have

$$(1 - \lambda)t + \lambda - t^{1-\lambda} \leq \lambda(1 - \lambda)(\log t)^2,$$

for $0 < t \leq 1$. Thus we have the following inequality for the invertible positive operator $mI \leq T \leq MI \leq I$:

$$(1 - \lambda)T + \lambda - T^{1-\lambda} \leq \lambda(1 - \lambda) \max_{m \leq t \leq M} (\log t)^2.$$

Putting $T \equiv B^{-1/2}AB^{-1/2} \leq I$ (which satisfies $A \leq B$), then we have $\frac{1}{h} \leq B^{-1/2}AB^{-1/2} \leq h$, and then we have

$$(1 - \lambda)B^{-1/2}AB^{-1/2} + \lambda - \left(B^{-1/2}AB^{-1/2}\right)^{1-\lambda} \leq \lambda(1 - \lambda) \max_{\frac{1}{h} \leq t \leq h} (\log t)^2,$$

which implies the inequality (13), by multiplying $B^{1/2}$ to the both sides in the above inequality, since $A\sharp_{\lambda}B = B\sharp_{1-\lambda}A$.

■

Remark 2.4 It is natural to consider that our inequalities are better than Tominaga's inequalities under the assumption $A \leq B$. Firstly we compare our inequality (10) with (2). For this purpose we take two numerical example under the condition $0 < t \leq 1$.

- (i) Take $t = \frac{1}{2}$ and $\lambda = \frac{1}{20}$, then we have

$$\exp\left(\lambda(1 - \lambda)\left(1 - \frac{1}{t}\right)^2\right) - S(t) \simeq -0.0128295.$$

- (ii) Take $t = \frac{1}{2}$ and $\lambda = \frac{1}{10}$, then we have

$$\exp\left(\lambda(1 - \lambda)\left(1 - \frac{1}{t}\right)^2\right) - S(t) \simeq 0.0326986.$$

Thus we can conclude that there is no ordering between (10) and (2).

Secondly we compare our inequality (11) with (3). For this purpose we take two numerical example under the condition $0 < t \leq 1$.

(i) Take $t = \frac{1}{2}$ and $\lambda = \frac{1}{5}$, then we have

$$L(1, t) \log S(t) - \lambda(1 - \lambda) (\log t)^2 \simeq -0.0338368.$$

(ii) Take $t = \frac{1}{2}$ and $\lambda = \frac{1}{20}$, then we have

$$L(1, t) \log S(t) - \lambda(1 - \lambda) (\log t)^2 \simeq 0.0202141.$$

Thus we can conclude that there is no ordering between (11) and (3).

Therefore we may conclude our two reverse inequalities for Young's inequality do not trivial results under the assumption $A \leq B$.

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